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Engineering Notes

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Control of a Structure with Two Closely Spaced Frequencies

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Introduction

THIS Note explores the effectiveness and limitations of linear feedback control on a structure with closely spaced natural frequencies. Closed-form expressions are presented for the fundamental case of a structure with two modes controlled by a single force. Three algorithms are investigated: velocity feedback, pole allocation, and optimal control. Derivations of the results have been presented elsewhere.[1]

Gains for Three Control Algorithms

The equations for the closed-loop system are

$$\ddot{x}(t) + \Lambda x(t) = bu(t) \tag{1}$$

$$u(t) = -gx(t) - h\dot{x}(t) \tag{2}$$

Here, $\Lambda = \text{diag}\{\omega_{u1}^2, \omega_{u2}^2\}$, $g = 2\omega_a^2\{g_1, g_2\}$, and $h = 2\omega_a\{h_1, h_2\}$, where ω_{ui} are the open-loop natural frequencies and $\omega_a = (\omega_{u1} + \omega_{u2})/2$. The parameter $\beta = (\omega_{u2} - \omega_{u1})/(2\omega_a)$ gives the spacing of the natural frequencies. The natural frequencies are closely spaced if $\beta \ll 1$. Without loss of generality, it is assumed that $\beta \geq 0$. The eigenvalue problem is

$$\left[s_j^2 I + s_j bh + (\Lambda + bg)\right]\phi_j = 0 \tag{3}$$

where bh and bg are outer products of column and row vectors.

For velocity feedback control, the gains $b_i h_i = h$ and $g_i = 0$ yield equal values for the diagonal elements of the matrix bh in Eq. (3). The corresponding closed-loop damping ratios are[1] $\zeta_j = h \pm \text{Re}\sqrt{h^2 - \beta^2}$. When $h \leq \beta$, the radical is purely imaginary, and $\zeta_1 = \zeta_2 = h$. However, when $h > \beta$, the radical becomes purely real, and $\zeta_2 \to 0$ as h increases. Thus, ζ_2 has an upper bound β.

With pole allocation control, it is possible to make both damping ratios increase monotonically with the feedback gains. A natural scheme is to make the damping ratios equal. Velocity feedback control achieves the desired pole allocation when $h \leq \beta$. However, when $h > \beta$, it is necessary to use the following nonzero displacement feedback gains[1]: $-b_1 g_1 = b_2 g_2 = g = (h^2 - \beta^2)/(2\beta)$. The resulting closed-loop damping ratios are $\zeta_1 = \zeta_2 = h$. The displacement feedback gains become infinite when $\beta \to 0$, as expected from controllability theory.

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For optimal linear quadratic Gaussian (LQG) control, the gains are obtained from a 4×4 Riccati equation. Closed-form expressions were derived for the special important case of $b_1 = b_2$:

$$b_1 h_1 = b_2 h_2 = h = \sqrt{\frac{b_1^2}{4R\omega_a^2} + \frac{\beta}{2}\left(\sqrt{\beta^2 + \frac{b_1^2}{R\omega_a^2}} - \beta\right)} \tag{4}$$

and $-b_1 g_1 = b_2 g_2 = g = \sqrt{h^2 + \beta^2} - \beta$. The displacement gain g is expressed in terms of h to eliminate the explicit dependence on the weighting factor R of the quadratic performance index. This is convenient since the displacement gains for the other algorithms are in terms of h. The corresponding closed-loop damping ratios are[1]

$$\zeta_j = h \pm \text{Re}\sqrt{(\beta - \sqrt{h^2 + \beta^2})^2 - \beta^2} \tag{5}$$

When $h \leq \sqrt{3}\beta$, the outer radical is purely imaginary, and $\zeta_1 = \zeta_2 = h$. However, when $h > \sqrt{3}\beta$, the radical is real, and ζ_2 decreases. Unlike the velocity feedback case, ζ_2 approaches β instead of 0 as h increases and has an upper bound of $\sqrt{3}\beta$ instead of β.

The relationships between the closed-loop damping ratios and the velocity feedback gain h are shown in Fig. 1a. When $h \leq \beta$, all three algorithms yield identical results. However, when $h > \sqrt{3}\beta$, pole allocation provides the most damping to the system, and velocity feedback provides the least damping.

Response Index

To evaluate the effectiveness of the three control algorithms, the time integral of the structural energy is used as a response index:

$$E = (\pi\omega_a)^{-1}\int_0^\infty \text{tr}\left[\Phi^T(t)\Lambda\Phi(t) + \dot{\Phi}^T(t)\dot{\Phi}(t)\right] dt \tag{6}$$

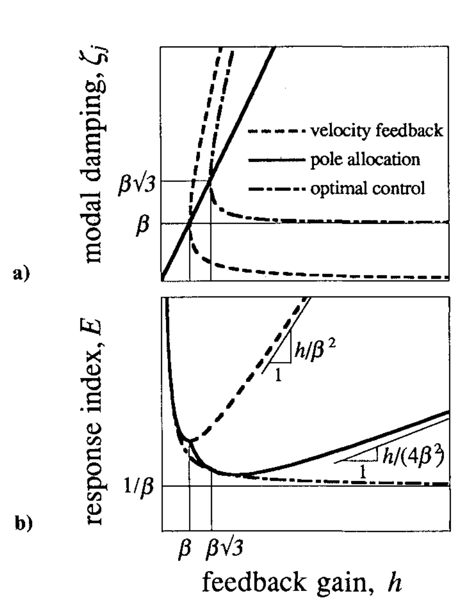

Fig. 1 Properties of the closed-loop system: a) modal damping ratios and b) response indexes.

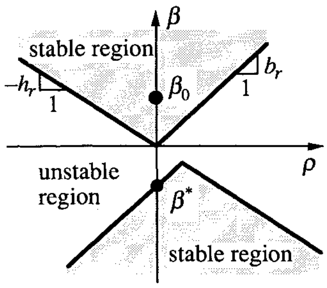

Fig. 2 Stability diagram.

where the various $\Phi(t)$ are the displacement components of the transition matrix for the closed-loop system. The trace indicates that an average is taken over all possible initial states satisfying $|x(0)|=1$ and $|\dot{x}(0)|=0$. Analytical expressions for E were developed for the case $b_1=b_2$; the results are[1] $E=(\beta^2+h^2)/(\beta^2 h)$ for velocity feedback and for pole allocation with $h\le\beta$, $E=(\beta^6+6h^2\beta^2+h^4)/(4\beta^2h^3)$ for pole allocation with $h>\beta$, and

$$E=\left(\frac{\beta}{h}+\frac{2h}{\beta}\right)\left(2\sqrt{h^2+\beta^2}-\beta\right)^{-1} \tag{7}$$

for optimal control. For all three algorithms, $E\to\infty$ as $\beta\to 0$, as expected.

Figure 1b shows how the response index varies with the feedback gain h. For $h<\beta$, the three algorithms are almost equally effective in reducing E. However, for larger gains, velocity feedback and pole allocation actually yield larger E as h increases. Only optimal control yields a response index that decreases monotonically with h. The asymptotic behavior of E for increasing h is[1] $E\sim h/\beta^2$ for velocity feedback, $E\sim h/4\beta^2$ for pole allocation, and $E\sim 1/\beta$ for optimal control. These results were illustrated by a numerical investigation of an antenna mast.[1]

Stability Conditions

If the open-loop parameters of the structure are not known with certainty, then the matrix Λ in Eq. (3) will have unknown frequency spacing β and off-diagonal elements $\rho\omega_a^2$. (Small uncertainties in ω_a are relatively unimportant.) Given a set of control parameters, it has been rigorously shown that the instability condition is[1]

$$\min\{b_r\rho, \beta^*+h_r\rho\}<\beta<\max\{b_r\rho, h_r\rho\} \tag{8}$$

when $\beta^*<0$ and

$$\min\{b_r\rho, h_r\rho\}<\beta<\max\{b_r\rho, \beta^*+h_r\rho\} \tag{9}$$

when $\beta^*\ge 0$. Here, $b_r=(b_1/b_2-b_2/b_1)/4$, $h_r=(h_1/h_2-h_2/h_1)/4$, and $\beta^*=(b_1h_1+b_2h_2)(g_1/h_1-g_2/h_2)/2$ are control-related parameters. (It is assumed that $b_1h_1>0$ and $b_2h_2>0$. This condition makes the diagonal elements of $\boldsymbol{bh}$ positive; it also assures that the average of the closed-loop modal damping ratios is positive.)

The shapes of the unstable and stable regions are shown in the $\rho-\beta$ space in Fig. 2 when $\beta^*<0$. The stable regions are two infinite, cone-shaped areas bounded by rays with slopes b_r and h_r. The point labeled β_0 corresponds to the assumed value for the structural parameters, i.e., the assumed model is $\rho=0$ and $\beta=\beta_0$. If the model is accurate, then the actual parameter value would differ only slightly from the assumed value, and the corresponding point in the $\rho-\beta$ space would be in a small neighborhood of β_0. Conversely, if the model is inaccurate, then the point corresponding to the actual parameter value may be quite far from β_0. The robustness of the structure with respect to modeling errors can be measured by the distance from β_0 to the boundaries of the unstable region. This distance is small if 1) the structure has very closely spaced natural frequencies, i.e., $|\beta_0|$ is small, and 2) either $|h_r|$ or $|b_r|$ is large. For either case, small modeling errors may result in instability. Although β^* governs the size of the unstable region, it does not affect the distance between β_0 and the boundary of the unstable region. These results were illustrated by a numerical investigation of the antenna mast.[1]

Conclusions

The paper has shown the following: 1) The normalized difference of the open-loop natural frequencies β governs the effectiveness and robustness of the control algorithms. 2) Velocity feedback, pole allocation, and optimal control yield responses that are nearly equal for small gains but are significantly different for velocity gains on the order of β or greater. Optimal control consistently yields the lowest responses. 3) The lower bound for the response of optimally controlled structures is on the order of $1/\beta$. Thus, structures with small β (i.e., closely spaced natural frequencies) cannot be effectively controlled by a single control input. 4) It is possible, through pole allocation, to increase the modal damping of the closed-loop system by increasing the feedback gains. However, when the velocity gains are larger than β, this algorithm is ineffective in controlling the response. 5) The stable region is bounded by two cones, with geometry determined by the control parameters. A system always becomes more robust as β increases.

Acknowledgments

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Reference

[1]Xu, K., Warnitchai, P., and Igusa, T., "Active Control of a Structure with Two Closely Space Natural Frequencies," *Proceedings of the AIAA/ASME/ASCE/AHS/ASC 33rd Structures, Structural Dynamics, and Materials Conference,* Vol. 3, AIAA, Washington, DC, 1992, pp. 1285–1295.

New Proof of the Jacobi Necessary Condition

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Introduction

LOOSELY speaking, a point along a reference extremal is called a conjugate point, if its state, time coordinate can be reached along a neighboring extremal with equal cost. As a typical example, consider the problem of finding the minimum

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